Quantum noise-induced chaotic oscillations

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We examine the weak quantum noise limit of the Wigner equation for phase space distribution functions. It has been shown that the leading order quantum noise described in terms of an auxiliary Hamiltonian manifests itself as an additional fluctuational degree of freedom which may induce chaotic and regular oscillations in a nonlinear oscillator.

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The absence of any direct counterpart to classical trajectories in phase space in quantum theory poses a special problem in nonlinear dynamical systems from the point of view of quantum-classical correspondence [1–3]. As an essential step towards understanding quantum systems a number of semiquantum methods, via Wentzel-Kramers-Brillouin (WKB) approximation, Ehrenfest theorem, or mean field approximation as well as some exact calculations, etc., have been proposed and investigated over the years [1–8]. A particularly noteworthy case [4] concerns a system that seems to be classically integrable but not in the quantum case due to tunneling. In the present paper we examine a related issue, i.e., the weak quantum noise limit of the Wigner equation for phase space distribution functions and show that it is possible to describe the quantum fluctuations of the system in terms of an auxiliary degree of freedom within an effective Hamiltonian formalism. This allows us to demonstrate an interesting quantum noise-induced chaotic and regular behavior in a driven double-well oscillator.

To start, we consider a one-degree-of-freedom system described by the Hamiltonian equation of motion:

$$\dot{x} = \frac{\partial H}{\partial p} = p,$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -V'(x,t), \tag{1}$$

where x and p are the coordinate and momentum variables for the system described by the Hamiltonian $H(x,p,t)$. $V(x,t)$ refers to the potential of the system. The reversible Liouville dynamics corresponding to Eq. (1) is given by

$$\frac{\partial \rho}{\partial t} = -p\frac{\partial \rho}{\partial x} + V'(x,t)\frac{\partial \rho}{\partial p}. \tag{2}$$

Here $\rho(x,p,t)$ is the classical phase space distribution function. For a quantum-mechanical system, however, x,p are not simultaneous observables because they become operators which obey the Heisenberg uncertainty relation. The quantum analog of classical phase space distribution function ρ corresponds to Wigner phase space function $W(x,p,t)$; x,p now being the c number variables. W is given by Wigner equation [9]:

$$\frac{\partial W}{\partial t} = -p\frac{\partial W}{\partial x} + V'(x,t)\frac{\partial W}{\partial p} + \sum_{n\geqslant 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}}. \tag{3}$$

The third term in Eq. (3) corresponds to quantum correction to classical Liouville dynamics.

Our aim in this report is to explore an auxiliary Hamiltonian description corresponding to Eq. (3) in the semiclassical limit $\hbar \to 0$. To put this in an appropriate context let us show below an analogy with an observation [10] on a weak thermal noise limit of overdamped Brownian motion of a particle in a force field.

In that significant analysis, Luchinsky and McClintock [10] have studied the large fluctuations (of the order $\gg \sqrt{D}, D$ being the diffusion coefficient) of the dynamical variables $\vec{x}$ away from and return to the stable state of the system with a clear demonstration of detailed balance. The physical situation is governed by the standard Fokker-Planck equation for probability density $P_c(\vec{x},t)$,

$$\frac{\partial P_c(\vec{x},t)}{\partial t} = -\vec{\nabla}\cdot\vec{K}(\vec{x},t)P_c(\vec{x},t) + \frac{D}{2}\nabla^2 P_c(\vec{x},t), \tag{4}$$

where $\vec{K}(\vec{x},t)$ denotes the force field.

In the weak noise limit D is considered to be a smallness parameter such that in the limit $D \to$ small, $P_c(\vec{x},t)$ can be described by a WKB-type approximation of the Fokker-Planck equation [10,11] of the form $P_c(\vec{x},t) = z(\vec{x},t)\exp[w(\vec{x},t)/D]$. Here $z(\vec{x},t)$ is a prefactor and $w(\vec{x},t)$ is the classical action satisfying the Hamilton-Jacobi equation which can be solved by the integration of an auxiliary Hamiltonian equation of motion [10]

$$\dot{\vec{x}} = \vec{p} + \vec{K}, \quad \dot{\vec{p}} = -\frac{\partial \vec{K}}{\partial \vec{x}}\vec{p},$$

$$H_{aux}(\vec{x},\vec{p},t) = \vec{p}\cdot\vec{K}(\vec{x},t) + \frac{1}{2}\vec{p}\cdot\vec{p}, \quad \vec{p} = \vec{\nabla}w, \tag{5}$$

where $\vec{p}$ is a momentum of the auxiliary system.

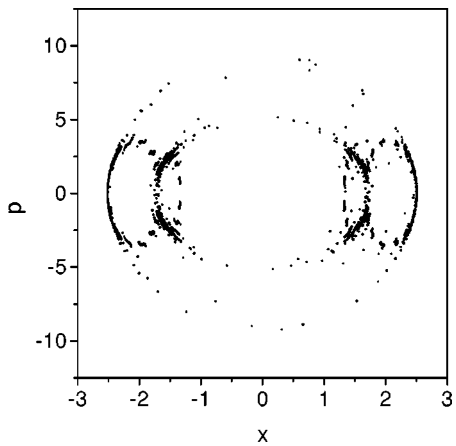

FIG. 1. Plot of x vs p on the Poincaré surface of section ($X=0$) for Eq. (11) with initial condition $x=-2.512, p=0, X\rightarrow 0, P=0$. (Units are arbitrary.)

The origin of this auxiliary momentum $\vec{p}$ is the fluctuations of the reservoir. In a thermally equilibrated system as emphasized by Luchinsky and McClintock [10], a typical large fluctuation of the variable $\vec{x}$ implies a temporary departure from its stable state $\vec{x}_s$ to some remote state $\vec{x}_f$ (in the presence of $\vec{p}$) followed by a return to $\vec{x}_s$ as a result of relaxation in the absence of fluctuations $\vec{p}$ (i.e., $\vec{p}=0$). Luchinsky and McClintock have studied these fluctuational and relaxational paths in analog electronic circuits and demonstrated the symmetry of growth and decay of classical fluctuations in equilibrium.

We now return to the present problem and in analogy to weak thermal noise limit we look for the weak quantum noise limit of Eq. (3) by setting $\hbar\rightarrow 0$ with $W(x,p,t)$ described by a WKB type approximation of the form

$$W(x,p,t)=W_0(x,t)\exp\left(-\frac{s(x,p,t)}{\hbar}\right), \tag{6}$$

where W_0 is again a preexponential factor and $s(x,p,t)$ is the classical action function satisfying Hamilton-Jacobi equation which can be solved by integrating the following Hamilton's equations:

$$\dot{x}=p,$$

$$\dot{X}=P,$$

$$\dot{p}=V'(x,t)-\sum_{n\geqslant 1}\frac{(-1)^{3n+1}}{2^{2n}}\frac{1}{(2n)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}X^{2n},$$

$$\dot{P}=V''(x,t)X-\sum_{n\geqslant 1}\frac{(-1)^{3n+1}}{2^{2n}(2n+1)!}\frac{\partial^{2(n+1)}V}{\partial x^{2(n+1)}}X^{2n+1} \tag{7}$$

with the auxiliary Hamiltonian H_{aux}

$$H_{aux}=pP-V'(x,t)X+\sum_{n\geqslant 1}\frac{(-1)^{3n+1}X^{2n+1}}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}, \tag{8}$$

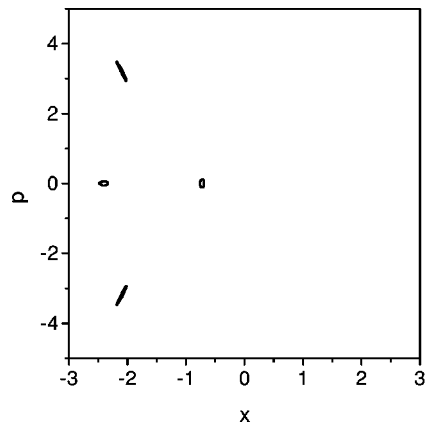

FIG. 2. Plot of x vs p for Eq. (1) with Hamiltonian (10) and initial condition $x=-2.512$ and $p=0.0$.

where we have defined the auxiliary coordinate X and momentum P as

$$X=\frac{\partial s}{\partial p} \quad \text{and} \quad P=\frac{\partial s}{\partial x}. \tag{9}$$

The interpretation of the auxiliary variables X and P is now derivable from the analysis of Luchinsky and McClintock [10]. The introduction of X and P in the dynamics implies the addition of a new degree of freedom into the classical system originally described by x,p. Since the auxiliary degree of freedom (X,P) owes its existence to the weak quantum noise, we must look for the influence of weak quantum fluctuations on the dynamics in the limit $X\rightarrow 0, P\rightarrow 0$, so that the Hamiltonian tends to be vanishing (since the X and P appear as multiplicative factors in the auxiliary Hamiltonian H_{aux}). It is therefore plausible that this vanishing Hamiltonian method captures the essential features of some generic quantum effect of the dynamics in classical terms in the weak quantum fluctuation limit. In what follows we shall be concerned with a quantum noise-induced barrier crossing dynamics—as a typical effect of this kind in a driven double-well system. Furthermore, since the auxiliary Hamiltonian

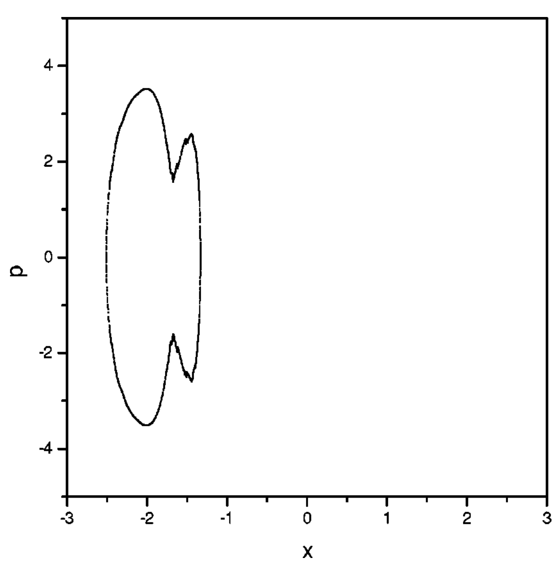

FIG. 3. Same as in Fig. 1 but for $x=-2.509$ and $p=0.0$.

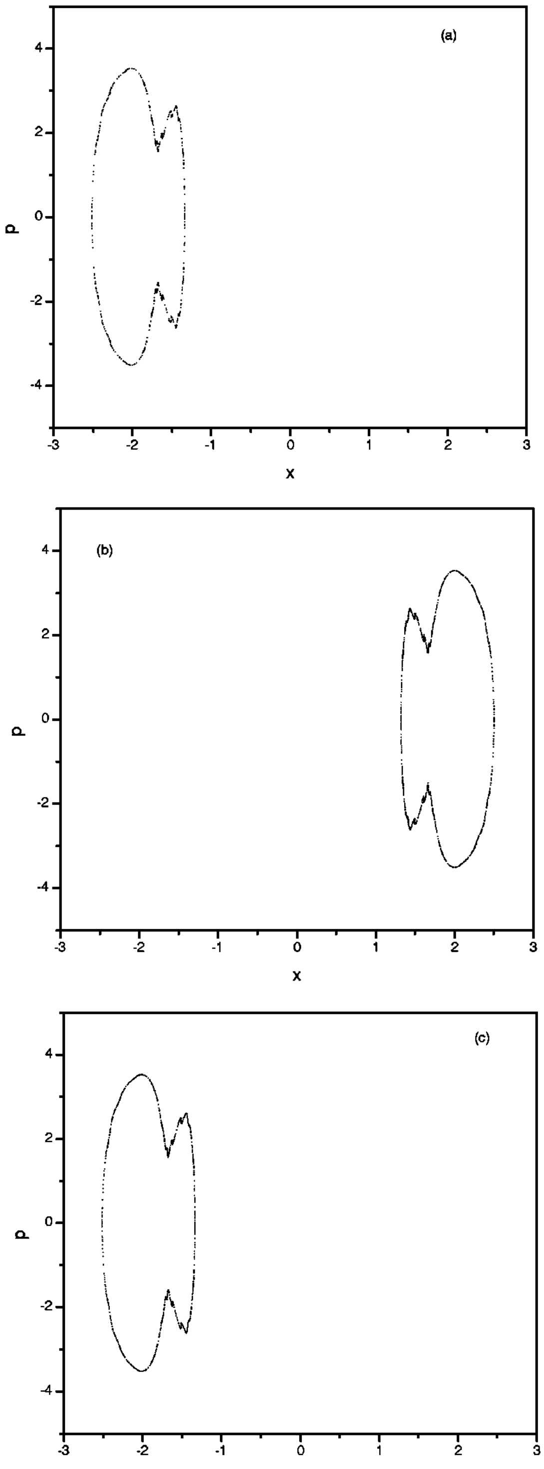

FIG. 4. Same as in Fig. 1 but for $x=-2.5093$ and $p=0$. The observations are taken for the time intervals (a) $t=0$ to $1293T$ (left well), (b) $t=1293T$ to $4291T$ (right well), and (c) $t=4291T$ to $7260T$ (left well) [T $(=2\pi/\Omega)$ is the time period of the driving field].

describes an effective two-degree-of-freedom system, the system, in general, by virtue of nonintegrability may admit chaotic behavior. This allows us to study a dynamical system where one of the degrees of freedom is of quantum origin. Thus if the driven one degree of freedom is chaotic, the influence of the quantum fluctuational degree of freedom on it appears to be quite significant from the point of view of what may be termed as quantum chaos. We point out, in passing, that the Wigner function approach of a somewhat different kind has also been considered earlier by Zurek and co-workers [8] for the analysis of the quantum decoherence problem in the context of the quantum-classical correspondence.

The testing ground of the above analysis is a driven double well oscillator characterized by the following Hamiltonian:

$$H=\frac{p^2}{2}+V(x,t),$$

$$V(x,t)=ax^4-bx^2+gx\cos\Omega t, \tag{10}$$

where a and b are the constants defining the potential. g includes the effect of coupling with the oscillator with the external field with frequency Ω. The model described by Eq. (10) has been the standard paradigm for studying chaotic dynamics over the last few years [12–15].

The equation of motion corresponding to the auxiliary Hamiltonian H_{aux} is given by

$$\dot{x}=p,$$

$$\dot{X}=P,$$

$$\dot{p}=4ax^3-2bx+g\cos\Omega t-3axX^2,$$

$$\dot{P}=(12ax^2-2b)X-aX^3. \tag{11}$$

In order to make the following numerical analysis consistent with this scheme of a weak quantum noise limit it is necessary to consider the limit of auxiliary Hamiltonian. To this end we fix the initial condition for the quantum noise degree of freedom $P=0$ and let $X\to$ very small for the entire analysis. The relevant parameters for the numerical study [14,15] are $a=0.5$, $b=10$, $g=10$, and $\Omega=6.07$.

The results of the numerical integration of Eq. (11) for the initial condition of the oscillator $p=0, x=-2.512$ (along with $P=0$ and $X=1.5\times10^{-6}$) are shown in the Poincaré plot (Fig. 1). What is apparent from a detailed follow-up of the system is that the system rapidly jumps back and forth between the two wells at irregular intervals of time resulting in a chaotic Poincaré map spread over the two wells. This is in sharp contrast to what we observe in Fig. 2 on plotting the results of the numerical integration of classical equations of motion corresponding to Eq. (1) and the Hamiltonian (10) with the same initial condition $p=0$ and $x=-2.512$. The system in this case resides in the four islands of the left well. It is thus immediately apparent that the quantum noise degree of freedom which imparts weak quantum fluctuations in the system through very small but nonzero X induces a passage from the left to right well and back.

In Fig. 3 we fix the initial condition at a different turning point $p=0, x=-2.509$ and calculate the auxiliary Hamiltonian dynamics Eq. (11). It is interesting to observe that the noise strength is not sufficient to make the system move from the left well where it stays permanently by depicting a closed regular curve on the Poincaré section.

The quantum noise-induced barrier crossing dynamics from the left to right well and back is illustrated in Figs. 4(a)–(c). The initial condition for the oscillator used in this case is $p=0, x=-2.5093$. The closed curve in Fig. 4(a) exhibits a snapshot of the confinement of the system (in the left well) upto the time $t=nT$ where $n=1293$ and T is the time period of the external field $(T=2\pi/\Omega)$. The system then jumps to the right well to stay there for a period of time $2998T$. This is shown in Fig. 4(b). The process goes on repeating for the next period of time $2969T$ when the system gets confined in the left well again. The back and forth quantum noise-induced oscillations between the two wells illustrate a regular dynamics in this case. In the absence of noise the classical system [Eq. (1)] remains localized in a specific well.

In summary, we have shown that the leading order quantum noise in the Wigner equation for phase space distribution functions results in an auxiliary Hamiltonian where the quantum noise manifests itself as an extra fluctuational degree of freedom. Depending on the initial conditions this may induce irregular or regular hopping between the two wells of a double-well oscillator. It is thus possible that a nonlinear system may sustain chaotic oscillations by quantum noise, even when its classical counterpart is fully regular.

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